$$u|_{\gamma_3} = f_1(s), \quad v|_{\gamma_4} = f_2(s), \quad f_1 \in H_{1/2}(\gamma_3), \quad f_2 \in H_{1/2}(\gamma_4)$$

where $k(s) \geqslant k_0 > 0$ is a piecewise-continuous function, k_0 is some constant, and $H_{1/2}(\gamma)$ is a Sobolev-Slobodetskii space.

Note 2. Using the explicit form of the operator K we can obtain sufficient conditions for the uniqueness of the solution, can study the differential properties of the solutions, and can also give a foundation for the Bubnov-Galerkin method.

BIBLIOGRAPHY

1. Vorovich, I. I., On the existence of solutions in nonlinear shell theory. Dokl. Akad. Nauk SSSR, Vol. 117, №2, 1957.
2. Mikhlin, S. G., Problem of the Minimum of a Quadratic Functional. Moscow-Leningrad, Gostekhizdat, 1952.
3. Sobolev, S. L., Some Applications of Functional Analysis in Mathematical Physics. Leningrad University Press, 1950.
4. Krasnosel'skii, M. A., Topological Methods in the Theory of Nonlinear Integral Equations. Moscow, Gostekhizdat, 1956.

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ASYMPTOTIC METHOD OF DETERMINING THE CRITICAL BUCKLING LOADS OF SHALLOW STRICTLY CONVEX SHELLS OF REVOLUTION

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An asymptotic method using the presence of a natural small parameter (the relative wall-thinness) is applied to determine the state of stress and strain of shallow strictly convex shells of revolution subjected to an axisymmetric load. In particular, asymptotic values of the upper and lower critical shell buckling loads are deduced under diverse boundary conditions and loading methods. An example of a spherical shell under uniform external pressure is examined. In the case of rigid clamping of the edge, the known result is obtained in [1] for the upper critical pressure. The values found for the upper critical pressures of spherical shells are in good agreement with the results of numerical computations on an electronic computer [2 – 13], and permit their continuation into the domain of arbitrarily thin shells where the machine computation is of low efficiency.

1. On the formulation of the problem. A system of nonlinear differential equations of axisymmetric deformation of shallow shells of revolution is considered [14]

$$\varepsilon^2 Av - {}^1/_2 u^2 + \theta u = 0, \qquad \varepsilon^2 Au + uv - \theta v + \varphi(r) = 0 \tag{1.1}$$

$$A(\ ) \equiv -r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r(\ ), \qquad \varphi(r) = \int_0^r q(t)\,t dt$$

$$\theta = \frac{\partial z}{\partial r}, \quad u = \frac{\partial w}{\partial r}, \quad v = \frac{\partial F}{\partial r}, \quad \left\{\frac{v}{r}, \frac{u}{r}\right\}_{r=0} < \infty$$

with different boundary conditions on the contour

$$\begin{aligned}
&1)\ v(1) = 0, \qquad \left[\frac{du}{dr} + \frac{\nu}{r}u\right]_{r=1} = 0 \\
&2)\ v(1) = 0, \qquad u(1) = 0 \\
&3)\ \left[\frac{dv}{dr} - \frac{\nu}{r}v\right]_{r=1} = 0, \qquad \left[\frac{du}{dr} + \frac{\nu}{r}u\right]_{r=1} = 0 \\
&4)\ \left[\frac{dv}{dr} - \frac{\nu}{r}v\right]_{r=1} = 0, \qquad u(1) = 0
\end{aligned} \tag{1.2}$$

All the quantities in (1.1), (1.2) are dimensionless and related to the dimensional quantities by means of the relationships

$$Z = az,\ W = aw,\ \xi = ar,\ \varepsilon^2 = h/a\gamma,\ \gamma^2 = 12(1 - \nu^2)$$
$$\Phi = Ea^2\varepsilon^2 F,\ p = E\gamma\varepsilon^4 q,\ \theta \leqslant -\alpha^2 r,\ 0 < \nu < 0.5$$

Here Z is the middle surface of the shell of revolution, W is the deflection, Φ is the Airy stress function, p is the intensity of the external load (the pressure) $p \geqslant 0$ and E is Young's modulus. The small parameter ε^2 characterizes the relative wall-thinness of the shell, h is the thickness, ν is the Poisson ratio. The direction ξ coincides with the direction of the exterior normal to the reference contour with radius a; $\alpha =$ const > 0.

The boundary conditions 1 – 4 in (1.2) correspond to: (1) movable hinge support of the shell edge, (2) sliding clamped edge, (3) fixed hinge support, (4) absolutely clamping of the edge.

For the case of a spherical shell under uniform external pressure, the problems (1.1), (1.2) have been studied extensively in order to investigate the state of stress and strain, and to determine the critical pressures. However, for sufficiently low values of $\varepsilon^2 = h / a\gamma$ (or equivalently, for high values of H / h, where H is the shell rise), the numerical methods applied converge weakly, and the machine computation is of low efficiency . Mathematically this results from the presence of the small parameter ε^2 in the highest derivatives in (1.1), and is explained from mechanics aspects by the fact that the edge effect phenomenon characterizing a sharp change in the stress resultants, moments, etc., originates in the neighborhood of the boundary (and in the operating zone of concentrated forces) of sufficiently thin shells.

An asymptotic method [15 – 18], based on the smallness of the parameter ε , and becoming the more exact the smaller the value of ε, is developed herein for the investigation of the state of stress and strain of shallow shells of revolution. Asymptotic expansions for the functions of the u- and v-solutions of problems (1.1), (1.2) are constructed, and simple formulas are also deduced for the asymptotic values of the upper and lower critical loads.

Let us define the upper critical load. Let $q = q(r, \sigma)$ depend smoothly on some parameter σ, which is determined by the behavior of the structure loading and which we shall call the load parameter. Let us assume that $q(r, 0) = 0$. It is well known that for small values of σ a continuous branch of the solutions $u(r, \sigma)$ and $v(r, \sigma)$ exists which is uniquely determined by the conditions $u(r, 0) = v(r, 0) = 0$. We call the greatest value of σ, for which the mentioned unique branch exists, the upper critical load σ^* of shell buckling. (Such a value σ^* is usually called the least bifurcation point). In other words, σ^* is the least value of the parameter σ for which another solution corresponding to a new equilibrium mode will appear in any sufficiently small neighborhood together with the main solution corresponding to the unbuckled equilibrium mode [1].

The parameter σ is introduced differently in different cases. In the case of positive pressure, which is considered herein, it is convenient to take a value proportional to an equivalent system of forces acting on the shell as σ, i.e. $\sigma = 2\varphi(1)$. In the case of a uniformly distributed external pressure $(q(r) \equiv q = \text{const})$, it is usually assumed that $\sigma = q$ [2 – 13]. Let us note that in the case of a sign-varying pressure, such an introduction of the parameter σ is inconvenient, and it is meaningless in the case of a self-equilibrating load $(\varphi(1) = 0)$.

Let $\varepsilon = 0$. We then have from (1.1)

$$-{}^1/_2 u_0^2 + \theta u_0 = 0, \quad u_0 v_0 - \theta v_0 + \varphi(r) = 0 \tag{1.3}$$

The system (1.3) has two solutions

$$\begin{aligned} &1)\ v_0 = \varphi\theta^{-1}, \quad u_0 = 0, \\ &2)\ v_0 = -\varphi\theta^{-1}, \quad u_0 = 2\theta \end{aligned} \tag{1.4}$$

which satisfy Eqs. (1.1) to accuracy of ε^2, but do not satisfy the boundary conditions (1.2). It is natural to expect that as $\varepsilon \to 0$ the problems (1.1), (1.2) have solutions which will behave just as (1.4) everywhere within the domain and will undergo strong changes such that boundary conditions (1.2) will be satisfied, only in the neighborhood of the point $r = 1$. These changes are described by functions of boundary layer type.

2. Construction of the asymptotics. Asympotic expansions of the solution of each of the problems (1.1), (1.2) are constructed for $\varepsilon \to 0$ as

$$v \sim v_\varepsilon \equiv \sum_{i=0}^{n} \varepsilon^i [v_i(r) + h_i(r, \varepsilon)], \qquad u \sim u_\varepsilon \equiv \sum_{i=0}^{n} \varepsilon^i [u_i(r) + g_i(r, \varepsilon)] \tag{2.1}$$

The functions v_i and u_i are obtained by using a first iteration process analogously to Sect. 2 in [17]. We hence obtain a system (1.3) to determine v_0 and u_0, and the following system to determine v_i and u_i $(i \geqslant 1)$

$$\theta u_i - \sum_{k+j=i} u_k u_j + A v_{i-2} = 0, \qquad \sum_{k+j=i} u_k v_j - \theta v_i + A u_{i-2} = 0 \quad (u_{-1} = v_{-1} \equiv 0) \tag{2.2}$$

Asymptotic expansions are constructed in the neighborhood of the first solution (1.4) to find the solution corresponding to the equilibrium mode in the subcritical stage. The boundary layer functions h_i and g_i are concentrated in the neighborhood of the boundary $r = 1$. They describe the behavior of the shell in the edge effect zone and are sought by using the second iteration process. Proceeding just as in [17], we obtain a system of nonlinear differential equations for h_0 and g_0

$$h_0'' + {}^1/_2 g_0^2 - \theta_0 g_0 = 0, \qquad g_0'' - g_0 h_0 + \theta_0 h_0 - \frac{\varphi_0}{\theta_0} g_0 = 0 \tag{2.3}$$

$$( )' = \frac{d( )}{dt}, \quad \theta_0 = \theta(1), \quad \varphi_0 = \varphi(1), \quad t = \frac{1-r}{\varepsilon}, \quad \{h_0, g_0\}_\infty \to 0$$

with the boundary conditions corresponding to (1.2)

$$\begin{aligned}
&1)\ h_0(0) = \varphi_0\theta_0^{-1},\ g_0'(0) = 0,\\
&2)\ h_0(0) = \varphi_0\theta^{-1}{}_0,\ g_0(0) = 0\\
&3)\ h_0'(0) = 0,\ g_0'(0) = 0,\\
&4)\ h_0'(0) = 0,\ g_0(0) = 0
\end{aligned} \tag{2.4}$$

Let us note that the boundary conditions at infinity result from the requirement of decreasing boundary layer functions.

We obtain systems of linear differential equations to determine h_i and g_i. In the case of boundary conditions 1 and 2 in (1.2), we have

$$h_i'' + g_i(g_0 - \theta_0) = th_{i-1}'' + h_{i-1}' + \sum_{k+j+2=i} t^k h_j - {}^1/_2 \sum_{m+n=i} g_m g_n +$$
$$+ \sum_{l+p=i} \theta_l t^l g_p - \sum_{k+l+j+2=i} u_{kl} t^l g_j \equiv F_{i1}, \quad g_i'' - h_i(g_0 - \theta_0) - g_i h_0 -$$
$$- v_0(1) g_i = tg_{i-1}'' + g_{i-1}' + \sum_{k+j+2=i} t^k g_j + \sum_{m+n=i} h_m g_n + \sum_{m+l+p=i} u_{ml} t^l h_p +$$
$$+ \sum_{m+l+p=i} v_{ml} t^l g_p \equiv F_{i2}; \qquad m \neq 0, \quad n \neq 0, \quad p \neq i \quad \{h_i, g_i\}_\infty \to 0 \tag{2.5}$$
$$\{\theta_l, u_{ml}, v_{ml}\} = \frac{1}{l!} \frac{\partial^l}{\partial r^l} \{\theta, u_m, v_m\}|_{r=1}, \qquad l = 0, 1, 2, \ldots$$

with the appropriate boundary conditions

$$\begin{aligned}
&1)\ h_i(0) = -v_i|_{r=1}, \qquad g_i'(0) = \left[\frac{du_{i-1}}{dr} + \nu u_{i-1}\right]_{r=1} + \nu g_{i-1}(0)\\
&2)\ h_i(0) = -v_i|_{r=1}, \qquad g_i(0) = -u_i|_{r=1}
\end{aligned} \tag{2.6}$$

In the case of boundary conditions 3 and 4 in (1.2), it follows from (2.3), condition 3 in (2.4) and (2.3), condition 4 in (2.4) that $h_0 = g_0 \equiv 0$. We then arrive at systems of linear differential equations with constant coefficients to determine h_i and g_i $(i \geqslant 1)$

$$h_i'' - \theta_0 g_i = f_{i1}, \qquad g_i'' + \theta_0 h_i - \frac{\varphi_0}{\theta_0} g_i = f_{i2} \tag{2.7}$$

with the conditions $\{h_i, g_i\} \to 0$ as $t \to \infty$ and the boundary conditions for $t = 0$ in the case 3 and 4 in (1.2), respectively

$$h_i'(0) = \left[\frac{dv_{i-1}}{dr} - \nu v_{i-1}\right]_{r=1} - \nu h_{i-1}(0), \qquad g_i'(0) =$$
$$= \left[\frac{du_{i-1}}{dr} + \nu u_{i-1}\right]_{r=1} + \nu g_{i-1}(0) \tag{2.8}$$
$$h_i'(0) = \left[\frac{dv_{i-1}}{dr} - \nu v_{i-1}\right]_{r=1} - \nu h_{i-1}(0), \qquad g_i(0) = -u_i|_{r=1}$$

Here f_{i1} and f_{i2} agree with F_{i_1} and F_{i_2}, respectively, from (2.5) if $h_0 = g_0 \equiv 0$. is substituted in the latter. In particular, we find $f_{11} = f_{12} = 0$ for $i = 1$. Then we obtain for the principal terms of h_1 and g_1 in (2.1) in cases 3 and 4 in (1.2), respectively

$$h_1\left(\frac{1-r}{\varepsilon}\right)=\frac{B}{2ab}\left[a\left(1+\frac{Q}{2}\right)x+b\left(1+\frac{Q}{2}\right)y\right],\quad g_1\left(\frac{1-r}{\varepsilon}\right)=$$
$$=\frac{B}{2ab}\,[ax+by]$$
$$h_1\left(\frac{1-r}{\varepsilon}\right)=\frac{B}{b}\left[\frac{Q}{4}x-2aby\right],\qquad g_1\left(\frac{1-r}{\varepsilon}\right)=\frac{B}{b}x \tag{2.9}$$

Here

$$x=e^{-a\tau}\sin b\tau,\quad y=e^{-a\tau}\cos b\tau,\quad \tau=\frac{1-r}{\varepsilon\sqrt{-\theta_0}},\quad Q=\frac{2\varphi_0}{\theta_0^2}$$
$$B=[-\theta_0]^{3/2}\,[(\varphi\theta^{-1})'-\nu\varphi\theta^{-1}]_{r=1},\quad a=\left(\frac{4-Q}{8}\right)^{1/2},\quad b=\left(\frac{4+Q}{8}\right)^{1/2}$$

Evidently (2.9) are valid only for $\varphi_0<2\theta_0^2$.

3. Solution of the edge effect equation (2.3). Making the substitution

$$h_0=-\theta_0 h,\quad g_0=-\theta_0 g,\quad \varphi_0=\frac{Q}{2}\theta_0^2,\quad t=(-\theta_0)^{1/2}\tau$$

we obtain from (2.3), (2.4)

$$\frac{d^2h}{d\tau^2}+\frac{1}{2}g^2+g=0,\quad \frac{d^2g}{d\tau^2}-gh-h+\frac{Q}{2}g=0,\quad \{h,g\}_\infty\to 0 \tag{3.1}$$

with the boundary conditions

$$\begin{aligned}&1)\ h|_{\tau=0}=\frac{Q}{2},\quad \left.\frac{dg}{d\tau}\right|_{\tau=0}=0\\&2)\ h|_{\tau=0}=\frac{Q}{2},\quad g|_{\tau=0}=0\end{aligned} \tag{3.2}$$

We reduce the solution of the problem (3.1), (3.2) to the solution of two nonlinear algebraic equations of infinite order by a method analogous to that elucidated (see [1], p. 52). To do this we seek the solution in the form

$$h=\sum_{k+m=1} a_{mk}e^{-a(m+k)\tau}x^my^k,\qquad g=\sum_{k+m=1} b_{mk}e^{-a(m+k)\tau}x^my^k$$
$$x=\sin b\tau,\quad y=\cos b\tau,\quad a=\left(\frac{4-Q}{8}\right)^{1/2},\quad b=\left(\frac{4+Q}{8}\right)^{1/2} \tag{3.3}$$

Substituting (3.3) into (3.1), and equating coefficients of x^my^k to zero after having reduced similar terms, we obtain a system of two equations to determine a_{01}, b_{01}, a_{10}, and b_{10}

$$b_{01}=\frac{Q}{4}a_{01}+2aba_{10},\qquad b_{10}=\frac{Q}{4}a_{10}-2aba_{01} \tag{3.4}$$

and a system of $2(n+1)$ linear algebraic equations to determine $a_{m,k}$ and $b_{m,k}$ ($m+k=n;\ m\geqslant 0,\ k\geqslant 0,\ n\geqslant 2$):

$$A_{mk}a_{m,k}+B_{mk}a_{m+1,k-1}+C_{mk}a_{m-1,k+1}+D_{mk}a_{m+2,k-2}+$$
$$+E_{mk}a_{m-2,k+2}+b_{mk}=-\frac{1}{2}\sum_{r+l=m,\,t+p=k} b_{rt}b_{lp} \tag{3.5}$$
$$A_{mk}b_{m,k}+B_{mk}b_{m+1,k-1}+C_{mk}b_{m-1,k+1}+D_{mk}b_{m+2,k-2}+$$
$$+E_{mk}b_{m-2,k+2}+\frac{Q}{2}b_{mk}-a_{mk}=\sum_{r+l=m,\,t+p=k} a_{rt}b_{lp}$$

Here

$$A_{mk} = a^2 (m + k)^2 - b^2 (2km + m + k), \quad B_{mk} = - 2ab (m + 1) (m + k)$$
$$C_{mk} = -B_{km}, \quad D_{mk} = b^2 (m + 2) (m + 1), \quad E_{mk} = b^2 (k + 2) (k + 1)$$

The system (3.5) is written in recurrent form and permits finding the unknowns a_{ij} and b_{ij} whose subscripts satisfy the condition $i + j = m + k$ if a_{ij} and b_{ij} with subscripts $i + j < m + k$ have already been found. We use the boundary conditions (3.2) to determine the unknowns a_{01} and a_{10}. Correspondingly, we obtain for cases 1 and 2 by using (3.3)

$$1) \sum_{m=1} a_{m0} = \frac{Q}{2}, \qquad a \sum_{k=1} kb_{0k} - b \sum_{k=0} b_{1k} = 0 \tag{3.6}$$
$$2) \sum_{m=1} a_{m0} = \frac{Q}{2}, \qquad \sum_{m=1} b_{m0} = 0$$

Here a_{mk} and b_{ij} $(m + k \geqslant 2, \; i + j \geqslant 1)$ are expressed in terms of a_{01} and a_{10} successively by means of (3.4) and (3.5). Hence, (3.6) are a system of two nonlinear algebraic equations of infinite order in a_{01} and a_{10}.

Let us seek the solution of problems (3.1), (3.2) by limiting ourselves to a finite number of terms in the sums (3.3), which results in the solution of nonlinear equations of finite order.

As ever in such cases, this number N is selected from the principle of practical convergence. Equations (3.4) – (3.6) corresponding to the given value of N, arc solved on an electronic computer by using an algorithm combining the method of continuation in the parameter Q and the Newton method. To do this (3.6) are written as

$$F (a_{01} (Q), \; a_{10} (Q), Q) = 0, \quad G (a_{01} (Q), \; a_{10} (Q), Q) = 0 \tag{3.7}$$

For $Q = 0$ the system (3.7) has the trivial solution $a_{01} = a_{10} = 0$. Let us set $Q = \Delta Q$. Then the zero solution is approximate if ΔQ is sufficiently small, and it is refined by the Newton method. Iteration is carried out for $Q = \Delta Q$ until a root is found with the given degree of accuracy. Afterwards, still another step is taken in Q and for $Q = 2\Delta Q$ the value obtained in the preceding stage is considered the approximate value of the root, etc. On approaching the bifurcation point Q^* the derivative $\partial a_{01} / \partial Q$ starts to grow strongly and then it is necessary to go over to motion on a_{01} or a_{10}, which permits the detrmination of Q^*.

A program in which automatic selection of the spacing associated with the number of iterations by the Newton method is provided, was compiled for the Odra-1204 electronic computer by using the algorithm mentioned. The values $Q^* = 0.793$ in the case of boundary conditions 1 in (3.2) and $Q^* = 1.766$ in the case of 2 in (3.2) were found. (The next digits are not written down although the values of the roots were sought to 10^{-9} accuracy for $N = 13$. The time required to obtain Q^* was 20 minutes).

The computations were checked in both cases by using the first integral

$$\frac{1}{2} (h'^2 - g'^2) + \frac{1}{2} g^2 h + gh - \frac{Q}{4} g^2 = 0, \qquad (\;)' = \frac{d(\;)}{d\tau} \tag{3.8}$$

which is obtained from (3.1) if the first equation is multiplied by h', and the second by $-g'$, they are added and integrated between τ and ∞. Assuming $\tau = 0$, and taking account of (3.2), we find the verifying formulas corresponding to cases 1 and 2 from

(3.8)

$$1)\quad h'^2(0) + Qg(0) = 0, \qquad 2)\quad h'^2(0) - g'^2(0) = 0 \tag{3.9}$$

Let us note that for the mentioned accuracy of the calculations in the neighborhood of Q^* the left sides of (3.9) yielded values between 2×10^{-5} and 5×10^{-5}.

Returning to the variables φ_0 and θ_0, we obtain values corresponding to the least bifurcation points

$$1)\quad {}^1/_2\sigma_0 \equiv \varphi_0 = 0.3965\,\theta^2(1), \qquad 2)\quad {}^1/_2\sigma_0 \equiv \varphi_0 = 0.883\,\theta^2(1) \tag{3.10}$$

4. Asymptotic value of the upper critical load. The result of Sect. 3 permits the asymptotic value of the upper critical load to be obtained at once in the case of a free edge support. This value can be refined if it is sought as a perturbation-theory series

$$\sigma^* \equiv 2\varphi^*(1) = \sigma_0 + \varepsilon\sigma_1 + \ldots + \varepsilon^n\sigma_n + \ldots \qquad \sigma_0 = 2\varphi_0(1) \tag{4.1}$$

We hence have for the corresponding function $\varphi(r)$

$$\varphi^*(r) = \varphi_0(r) + {}^1/_2(\varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \ldots + \varepsilon^n\sigma_n + \ldots)\,r^2$$

Then an additional member ${}^1/_2\,\sigma_i r^2$ will appear in the left side of Eq. (2.2) in constructing the asymptotic expansions (2.1), on which all the subsequent functions of the iteration processes will depend, starting with the number i. The values σ_i $(i \geqslant 1)$ are determined from the conditions for solvability of (2.6), (2.5). No numerical analysis is made here nor are terms sought for $i \geqslant 1$ in (2.1) and (4.1).

Writing (3.10) and (4.1) in dimensional variables, we arrive at the following result. Let $\varphi(r)\,\theta^{-1}(r)$ be a sufficiently smooth function for $0 \leqslant r \leqslant 1$. Then for sufficiently thin shells with free edge support, the values of the upper critical load P_n^* are determined from the formula

$$P_n^* \equiv 2\int_0^1 p_n^*(\xi)\,\xi d\xi = \frac{\alpha_n E h^2 a^{-2}\theta^2(1)}{\sqrt{3(1-\nu^2)}}\,[1 + a_{1n}\varepsilon + a_{2n}\varepsilon^2 + \ldots] \tag{4.2}$$

$$n = 1, 2, \quad \alpha_1 = 0.3965, \quad \alpha_2 = 0.883, \quad P = Eh^2\gamma^{-1}a^{-2}\theta^2(1)\sigma$$

The subscripts $n = 1, 2$ here correspond to boundary conditions 1 and 2 in (1.2). In the case of rigid clamping of the edge (boundary conditions 3 and 4 in (1.2)) the influence of the edge effect is considerably weaker and the principal term of the asymptotics in the edge effect zone is on the order of ε. It is determined uniquely from (2.9) for $\varphi(1) < 2\theta^2(1)$. For $\varphi(1) = 2\theta^2(1)$, Eqs. (2.7) have no decreasing solutions and the conditions at infinity are not satisfied.

In order to show that $\sigma_0 \equiv 2\varphi(1) = 4\theta^2(1)$ is an asymptotic value of the upper critical load in the case of rigid clamping of the shell edge, let us alter the process of constructing the asymptotics of Sect. 2 in the edge-effect zone somewhat. Namely, let us retain the method of obtaining the equations as before, but let us satisfy the appropriate boundary conditions exactly in each stage. This results in the determination of h_0 and g_0 from the system (2.3) with the appropriate boundary conditions for the cases 3 and 4 in (1.2)

$$3)\ h_0'(0)+\varepsilon\nu h_0(0)=\varepsilon\,[(\varphi\theta^{-1})'-\nu\varphi\theta^{-1}]_{r=1},\quad g_0'(0)-\varepsilon\nu g_0(0)=0$$
$$4)\ h_0'(0)+\varepsilon\nu h_0(0)=\varepsilon\,[(\varphi\theta^{-1})'-\nu\varphi\theta^{-1}]_{r=1},\quad g_0(0)=0 \qquad (4.3)$$

The problems(2.3) and (4.3) are solved exactly as in Sect. 3. The values of the least bifurcation points are obtained sufficiently close to $2\theta^2(1)$ and are the closer, the smaller the ε. The method of constructing the next terms of the asymptotics is the same as in the case of a free edge support. Therefore, for sufficiently thin shells with rigid clamping of the edge, the values of the upper critical loading P_n^* are determined from the formula

$$P_n^* \equiv 2\int_0^1 p_n^*(\xi)\,\xi d\xi = \frac{2Eh^2a^{-2}\theta^2(1)}{\sqrt{3(1-\nu^2)}}\,[1+a_{1n}\varepsilon+a_{2n}\varepsilon^2+\ldots] \qquad (4.4)$$

($n = 3, 4$. The coefficients a_{in} are not found here).

5. A spherical shell under uniform external pressure. In this case we must set

$$\varphi(r)=\frac{1}{2}qr^2,\quad \theta=-\lambda r,\quad \lambda=\frac{a}{R},\quad p=\frac{qE}{\sqrt{12(1-\nu^2)}}\left(\frac{h}{a}\right)^2$$

in formulas from Sect. 2 – 4. We then obtain for the values of the upper critical buckling pressure p_n^* of sufficiently thin shallow spherical shells from (4.2), (4.4)

$$p_n^* = \frac{\alpha_n E}{\sqrt{3(1-\nu^2)}}\left(\frac{h}{R}\right)^2[1+a_{1n}\varepsilon+a_{2n}\varepsilon^2+\ldots] \qquad (5.1)$$

$$n=1,2,3,4;\quad \alpha_1=0.3965,\quad \alpha_2=0.883,\quad \alpha_3=\alpha_4=2$$

Here, as in Sect. 4, the subscripts $n = 1, 2, 3, 4$ correspond to the boundary conditions 1 – 4 in (1.2). Values of p_n^* calculated taking account of just the first member in (5.1) were compared with the results of the upper critical pressures obtained on digital computers by various authors. The discrepancy did not exceed 7% for the least values of the parmeter $\varepsilon\lambda^{-1} \equiv h\,(2\gamma H)^{-1}$ presented in [4, 12] for $n = 1,2$. Formula (5.1) in the case of boundary conditions 3 and 4 in (1.2) was obtained earlier by a geometric method (see [1] p. 145). This is a well-known result of linear theory, however it is stressed in [1] that the formula must be applied for sufficiently thin shells. This result is verified for boundary conditions 4 in (1.2) by the results in [8] for $3.5\times10^{-3} < \varepsilon\lambda^{-1} < 5\times10^{-3}$ (the values of the upper critical pressures for lesser $\varepsilon\lambda^{-1}$ are not presented).

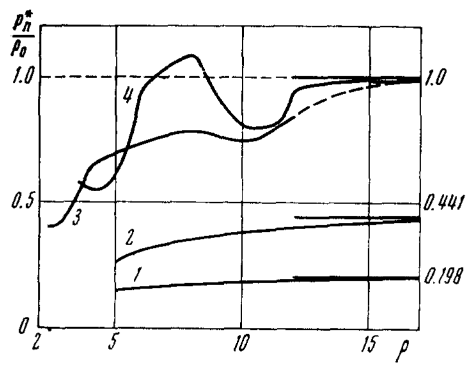

Fig. 1

In the case of conditions 3 in (1.2), on the basis of the data known to the author [12], it is still not possible to assess the behavior of p_3^* for $\varepsilon\lambda^{-1} \equiv h\,(2\gamma H)^{-1} > 7\times10^{-3}$ as $\varepsilon \to 0$. For this case the values of the upper critical pressure for $\varepsilon < \lambda/30$ are less than the corresponding values of p_1^*, and in order to emerge on the asymptotics it is necessary to carry out computations for lower values of ε. A dependence of the upper critical pressure on the geometric shell parameters is presented in Fig. 1 constructed by using the results of [4, 8, 12] as well as (5.1). The support conditions 1 – 4 in (1.2) correspond to curves *1–4* The value of the parameter p_n^*/p_0 is laid off along the vertical axis, and the value of the parameter ρ along the horizontal axis, where

$$\rho = \frac{2\gamma H}{h} = \frac{a^2\gamma}{Rh} = \frac{\lambda}{\varepsilon}, \qquad p_0 = \frac{4E}{\gamma}\left(\frac{h}{R}\right)^2, \qquad \gamma^2 = 12(1-\nu^2)$$

Further, a rigorous proof of (5.1) is given for p_3^*. In this connection, let us first prove the following.

Theorem. Let $\varepsilon \to 0$. Then for arbitrarily small $\delta > 0$ $(\delta = O(\varepsilon))$, there is a value ε_1 such that the dimensionless value of the upper critical pressure q^* satisfies the inequality $q^* \geqslant 4\lambda^2 - \delta$ for $0 < \varepsilon < \varepsilon_1$ in the problem (1.1), 3 in (1.2). Hence, for all $q < 4\lambda^2 - \delta$ in the neighborhood of v_ε, u_ε from (2.1) there exists just one solution, and the estimates

$$\max_r |v - v_\varepsilon| \leqslant m\varepsilon^n, \quad \max_r |u - u_\varepsilon| \leqslant m\varepsilon^n \qquad (0 \leqslant r \leqslant 1, \; n = 1, 2, \ldots) \tag{5.2}$$

are valid.

Proof. As follows from Sect. 2, the asymptotic expansion (2.1) can be represented in the case of the problem (1.1), 3 in (1.2), as

$$v_\varepsilon = -\frac{q}{2\lambda} r + \varepsilon s_1(r, \varepsilon), \qquad u_\varepsilon = \varepsilon s_2(r, \varepsilon) \tag{5.3}$$

$$s_1(r, \varepsilon) = \sum_{i=1}^{n} \varepsilon^{i-1}(h_i + \alpha_i) + \varepsilon^n \varphi_1, \qquad s_2(r, \varepsilon) = \sum_{i=1}^{n} \varepsilon^{i-1}(g_i + \beta_i) + \varepsilon^n \varphi_2$$

Here h_i and g_i are defined in Sect. 2, α_i and β_i are constructed exactly as in Sect. 2 of [17], and the sufficiently smooth arbitrary functions $\varphi_1(r)$ and $\varphi_2(r)$ correspond to the single requirement that the right sides of (5.3) must satisfy the boundary conditions 3 in (1.2) exactly.

Let us first prove the validity of the asymptotic expansion for $q < 4\lambda^2 - \delta$. To do this, let us use Theorem 4.2 in [18]. The method of obtaining the a priori estimates needed for compliance with the conditions of this theorem is analogous to [17]. Considering the problem (1.1), 3 in (1.2) as a functional equation $\mathbf{P}(\mathbf{V}) = 0$, as in [17], we obtain the estimates

$$\|\mathbf{P}(\mathbf{V}_\varepsilon)\|_{L_2} \leqslant c_1\varepsilon^{n+1}, \qquad \|\mathbf{P}''\| \leqslant c_3, \qquad \mathbf{V}_\varepsilon \equiv \left(\frac{-qr}{2\lambda} + \varepsilon s_1, \; \varepsilon s_2\right) \tag{5.4}$$

It is somewhat more difficult to obtain the estimate

$$\|[\mathbf{P}'_{\mathbf{V}_\varepsilon}]^{-1}\| \leqslant c_2\varepsilon^{-4}, \quad \mathbf{P}'_{\mathbf{V}_\varepsilon}(\mathbf{V}) \equiv \left\{\varepsilon^2 Av - (\varepsilon s_1 + \lambda r)u, \quad \varepsilon^2 Au - \frac{qr}{2\lambda}u + \varepsilon s_2 u + \varepsilon s_1 v + \lambda rv\right\} \tag{5.5}$$

Here $\mathbf{P}'_{V_\varepsilon}$ is the Fréchet derivative on the element $\mathbf{V}_\varepsilon$.

Let us consider the system of equations

$$\mathbf{P}'_{\mathbf{V}_\varepsilon}(\mathbf{V}) = \mathbf{f}, \qquad \mathbf{f} \equiv (f_1, f_2), \qquad \mathbf{V} \equiv (v, u) \tag{5.6}$$

We multiply the first equation in (5.6) by $(\delta_1 v - u)$, and the second by $\delta_1 v + u$, where δ_1 is some small positive number, and we integrate between zero and one and add. We consequently obtain

$$\delta_1\varepsilon^2\left[\int_0^1\left(rv'^2 + \frac{v^2}{r} + ru'^2 + \frac{u^2}{r}\right)dr + \nu u^2(1) - \nu v^2(1)\right] - \frac{q\delta_1}{2\lambda}\int_0^1 ru^2 dr +$$

$$+\lambda\int_0^1 r(u^2+v^2)\,dr-\frac{q}{2\lambda}\int_0^1 ruv\,dr+\varepsilon\int_0^1(\delta_1 s_2u^2+s_2uv+s_1v^2+s_1uv)\,dr+$$

$$+2\nu\varepsilon^2u(1)v(1)=\int_0^1[(\delta_1f_1+f_2)v+(\delta_1f_2-f_1)u]\,dr \tag{5.7}$$

Let us note the following inequalities:

$$\varepsilon\int_0^1(\delta_1s_2u^2+s_2uv+s_1v^2+s_1uv)\,dr\leqslant\delta_2\int_0^1 r(u^2+v^2)\,dr$$

$$\delta_2=O(\varepsilon),\quad 2v(1)u(1)\leqslant v^2(1)+u^2(1),\quad v^2(1)\leqslant a^2\int_0^1 rv'^2dr+(2+a^{-2})\int_0^1 rv^2dr \tag{5.8}$$

The first two inequalities are evident, but the third follows from the chain of relationships

$$v^2(1)=[rv(r)]^2_{r=1}=2\int_0^1(r^2v'v+rv^2)\,dr\leqslant a^2\int_0^1 r^2v'^2dr+$$

$$+\frac{1}{a^2}\int_0^1 r^2v^2dr+2\int_0^1 rv^2dr\leqslant a^2\int_0^1 rv'^2dr+\left(2+\frac{1}{a^2}\right)\int_0^1 rv^2dr$$

Now, setting $a^2=1/2\,\nu^{-1}\delta_1(1+\delta_1)^{-1}$, we deduce from (5.7) using (5.8)

$$\left[\lambda-\delta_2-\varepsilon^2(1+\delta_1)2\nu-2\varepsilon^2\nu^2(1+\delta_1)^2\delta_1^{-1}-\frac{q\delta_1}{2\lambda}\right]\int_0^1 r(u^2+v^2)\,dr+$$

$$+\frac{\delta_1\varepsilon^2}{2}\int_0^1\left(rv'^2+\frac{v^2}{r}+ru'^2+\frac{u^2}{r}\right)dr-\frac{q}{2\lambda}\int_0^1 ruv\,dr\leqslant\sqrt{2}\,\|\mathbf{f}\|_{L_2}\|\mathbf{V}\|_{L_2} \tag{5.9}$$

Let $\delta_1=\varepsilon$ and $q<4\lambda(\lambda-2K\varepsilon)$, where $K>0$ and

$$\left|\delta_2-\varepsilon^2(1+\varepsilon)2\nu-2\varepsilon\nu^2(1+\varepsilon)^2-\frac{q\varepsilon}{2\lambda}\right|\leqslant\frac{K\varepsilon}{2}$$

We then find the following estimates from (5.9)

$$\|\mathbf{V}\|_{L_2}\leqslant 2^{3/2}(\varepsilon K)^{-1}\|\mathbf{f}\|_{L_2},\qquad\int_0^1\left(rv'^2+\frac{v^2}{r}+ru'^2+\frac{u^2}{r}\right)dr\leqslant 8\varepsilon^{-4}K\|\mathbf{f}\|^2_{L_2}$$

Hence, we have in the interval $0\leqslant r\leqslant 1$

$$\max_r|v|+\max_r|u|\leqslant m_3\varepsilon^{-2}$$

and the estimate (5.5) is obtained from (5.6). By using (5.4) and (5.5) we verify that all the conditions of Theorem 4.2 from [18] are satisfied if $n>7$ and ε is sufficiently small. The estimates (5.2) are now proved by using the triangle inequality.

Therefore, as $\varepsilon\to 0$ and $q<4\lambda(\lambda-2K\varepsilon)$ there exists a solution of the problem (1.1), 3 from (1.2) for which the asymptotic expansions (5.3) are valid. Let us note that the L.V.Kantorovich theorem on the convergence of the Newton operator method, from which Theorem 4.2 from [18] has been obtained, assures the uniqueness of this solution in the neighborhood of (5.3). This latter results in the deduction that as $\varepsilon\to 0$ the value of the upper critical pressure is $q^*\geqslant 4\lambda^2-\delta$, where $\delta=O(\varepsilon)$. The theorem is proved.

Furthermore, let us note that for $q \leqslant 4\lambda^2 - \delta$, where $\delta = O(\varepsilon)$ as $\varepsilon \to 0$, the boundary values v_ε and u_ε from (5.3), and therefore the solution of the problem (1.1), 3 in (1.2) as well, tend, respectively, to the following values:

$$v|_{r=1} = -\frac{q}{2\lambda}, \qquad u|_{r=1} = 0, \qquad \left\{\frac{v}{r}, \frac{u}{r}\right\}_{r=0} < \infty \tag{5.10}$$

For any value of q the problem (1.1), (5.10) has the trivial solution $v = -1/2\, q r \lambda^{-1}$, $u=0$ corresponding to the membrane equilibrium mode. It has been shown in [20, 19] that from one to three new solutions of the problem (1.1), (5.10) can appear in the left semicircle of the least eigenvalue of the corresponding linearized boundary value problem $(4\lambda^2 - \mu, \mu)$, where μ is some small positive number. Therefore, $q^* = 4\lambda^2$ is a bifurcation point as $\varepsilon \to 0$.

6. On the asymptotic value of the lower critical load. To find the solution corresponding to the mirror buckled equilibrium mode, the asymptotic expansions (2.1) are constructed in the neighborhood of the second solution of (1.4). To determine h_0 and g_0 we obtain a system of edge effect equations

$$h_0'' + {}^1/_2 g_0^2 + \theta_0 g_0 = 0, \; g_0'' - g_0 h_0 - \theta_0 h_0 - \frac{\varphi_0}{\theta_0} g_0 = 0, \; \{h_0, g_0\}_\infty \to 0 \tag{6.1}$$

with boundary conditions corresponding to (1.2)

1) $h_0(0) = -\varphi_0\theta_0^{-1}, g_0'(0) = 0,$

2) $h_0(0) = -\varphi_0\theta_0^{-1}, \; g_0(0) = -2\theta_0$

3) $h_0'(0) = 0, \; g_0'(0) = 0, \quad$ 4) $h_0'(0) = 0, \; g_0(0) = -2\theta_0 \qquad (6.2)$

The subsequent terms of the asymptotics are determined exactly as in Sect. 2. Solving the problem (6.1), (6.2) by the method in Sect. 3, and using reasoning analogous to Sect. 4, we arrive at the following result.

Let $\varphi(r)\,\theta^{-1}(r)$ be a sufficiently smooth function for $0 \leqslant r \leqslant 1$. Then for sufficiently smooth thin shells, the value of the lower critical load P_{*n} is determined by the formula

$$P_{*n} \equiv 2\int_0^a p_{*n}(\xi)\,\xi d\xi = \frac{\beta_n E h^2 a^{-2}\theta^2(1)}{\sqrt{3(1-\nu^2)}}\,[1 + b_{1n}\varepsilon + \ldots]$$

$$\beta_1 = -0.3965, \quad \beta_2 = -0.057, \quad \beta_3 = -2, \quad \beta_4 = 2.58\cdot 10^{-3}$$

The subscripts $n = 1, 2, 3, 4$ correspond to the boundary conditions 1 – 4 in (1.2). Here the lower critical load is defined as the least value $\sigma_* \equiv 2\varphi_*(1)$ at which a continuous unique branch of the solution $u(r, \sigma)$ and $v(r, \sigma)$ corresponding to the mirror buckled equilibrium mode exists. Let us note that this definition is not completely rigorous since there is no proof of the fact that σ_* is the value of the least load perceivable by the shell [1].

Therefore, in the first three edge support cases the value of the lower critical load is negative, and only in the case of absolutely clamped edge it is positive. In the case of a hinge supported edge (boundary conditions 1 and 3 in (1.2)), this fact has a rigorous mathematical foundation [17, 21].

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BIBLIOGRAPHY

1. Pogorelov, A.V., Geometric Methods in the Nonlinear Theory of Elastic Shells. Moscow, "Nauka", 1967.
2. Budiansky, B., Buckling of clamped shallow spherical shells. Proc. Symp. Theory of Thin Elastic Shells, Delft, 1959. Amsterdam, North-Holland Publ. Co., 1960.
3. Keller, H.B. and Reiss, E.L., Spherical cap snapping. J. Aero/Space Sci., Vol. 26, №10, 1959.
4. Weinitchke, H.J., On the stability problem for shallow spherical shells. J. Math. and Phys., Vol. 38, №4, 1960.
5. Murray, F.J. and Wright, F.W., The buckling of thin spherical shells. J. Aero/Space Sci., Vol. 28, №3, 1961.
6. Thurston, G.A., A numerical solution of the nonlinear equations for axisymmetric bending of shallow spherical shells. Trans. ASME, Ser. E, J. Appl. Mech., Vol. 28, №4,.
7. Fedos'ev, V.I., On a method of solution of the nonlinear problems of stability of deformable systems. PMM Vol. 27, №2, 1963.
8. Keller, H.B. and Wolfe, A.W., On the nonunique equilibrium states and buckling mechanism of spherical shells. J. Soc. Indust. and Appl. Math., Vol. 13, №3, 1965.
9. Vorovich, I.I. and Zipalova, V.F., On the solution of nonlinear boundary value problems of the theory of elasticity by a method of transformation to an initial value Cauchy problem. PMM Vol. 29, №5, 1965.
10. Vorovich, I.I. and Zipalova, V.F., Analysis of the nonlinear strain of a spherical dome in high approximations. Inzh. Zh. Mekh. Tverd. Tela, № 2, 1966.
11. Valishvili, N.V. and Stegnii, V.N., On equilibrium modes of shallow spherical shells. Inzh. Zh., Mekh. Tverd. Tela, №6, 1968.
12. Valishvili, N.V., On the stability of shallow spherical shells. Trudy, 7th All-Union Conf. on Plate and Shell Theory. Dnepropetrovsk, 1969. Moscow, "Nauka", 1970.
13. Shil'krut, D.I., Shevandronov, N.V., Morar, V.P. and Maksimov, Iu.A., Solution of Nonlinear Shell Theory Problems on Analog Computers by the Direct Modeling Method and on Digital Electronic Computers by the Method of Direct Reduction to a Cauchy Problem. Kishinev, Akad. Nauk Mold. SSR Press, 1969.
14. Feodos'ev, V.I., Elastic Elements of Precise Instrumentation. Moscow, Oborongiz, 1949.
15. Srubshchik, L.S., Asymptotics of the Reissner equations in the nonlinear theory of symmetrically loaded shells of revolution. Dokl. Akad. Nauk SSSR, Vol. 182, №3, 1968.
16. Vorovich, I.I. and Srubshchik, L.S., Asymptotic analysis of the general equations of nonlinear shallow shell theory. Trudy, 7th All-Union Conf. on Plate and Shell Theory. Dnepropetrovsk, 1969. Moscow, "Nauka", 1970.
17. Srubshchik, L.S., Nonstiffness of spherical shells. PMM Vol. 31, №4, 1967.

18. Srubshchik, L. S. and Iudovich, V. I., Asymptotic integration of the system of equations for the large deflections of symmetrically loaded shells of revolution. PMM Vol. 26, №5, 1962.
19. Vorovich, I. I., Certain questions of shell stability in the large. Dokl. Akad. Nauk SSSR, Vol. 122, №1, 1958.
20. Reiss, E. L., Bifurcation buckling of spherical caps. Comm. Pure and Appl. Math., Vol. 18, №1/2, 1965.
21. Srubshchik, L. S., On the question of nonstiffness in nonlinear shallow shell theory. Izv. Akad. Nauk SSSR, Ser. Mat., №3, 1972.

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ON OPTIMAL THICKNESS OF A CYLINDRICAL SHELL LOADED BY EXTERNAL PRESSURE

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The problem of an optimal, from the weight viewpoint, thickness distribution law along the length of a cylindrical shell loaded by axisymmetric external pressure is examined when collapse occurs because of buckling. The apparatus of the generalized maximum principle is used for the solution [1].

1. The problem is formulated in the terminology of the theory of optimal processes. The state of the shell during loading is given by the phase coordinates φ_j ($j = 1,2, \ldots, 6$) at each instant α, where α is the dimensionless length coordinate. A change in the phase coordinates, as α changes, corresponds to shell motion. This process can be controlled by changing the shell thickness $\delta(\alpha)$. The highest derivative $\delta^n(\alpha)$ [2] in the motion (stability) equation is taken as the control function, and the functions $\delta(\alpha), \ldots, \delta^{n-1}(\alpha)$ as the phase coordinates. The problem is to seek a function $\delta(\alpha)$ satisfying the stability equations, as well as boundary conditions and constraints, such that the minimum of the quantity

$$J = \int_0^{L/R} \delta(\alpha)\, d\alpha$$

would be achieved. Here R and L are the shell radius and length, respectively.

Constraints are imposed from structural or engineering considerations, as well as from the strength condition $\delta(\alpha) \geqslant \delta_{\min}$ and the additional condition associated with the selected model of shell analysis $\delta^n(\alpha) \leqslant a$. An optimal shell is sought in the class of admissible shells, which can be computed by using the Kirchhoff-Love hypothesis. Hence, the stability equations of classical shell theory are taken as the trajectory equations, and a constraint from the condition [3]

$$\frac{1}{R}\frac{d\delta(\alpha)}{d\alpha} \leqslant \frac{\delta_{\max}}{R} \tag{1.1}$$